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Undecidability of Winkler's r-Neighborhood Problem for Covering Digraphs

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Consider the following decision problem which P. Winkler and V. Bulitko (independently) showed was undecidable: Given a finite set Φ of rooted graphs and a positive integer r, is there a graph G such that Φ represents, up to isomorphism, the set of all r-neighborhoods of G? We show the undecidability of the related problem in which G is required to be the covering digraph of a partial ordering. Our construction shows that the problem remains undecidable (for certain fixed r) even when G is also required to be planar and bipartite.

1. INTRODUCTION

All graphs and digraphs in this paper are finite. Recall that for vertices u and v in the same connected component of a graph $G=(V, E)$, the *distance* from u to v is the minimum number of edges in a path from u to v. When G has directed edges the distance between two vertices is their distance in the underlying undirected graph. For a positive integer r, the *r-neighborhood* of v, denoted $\mathcal{N}_r(v)$, is the subgraph of G, having distinguished vertex v, induced by all vertices of distance at most r from v. The *r-neighborhood set* of a graph (or digraph) G, denoted $\mathcal{N}_r(G)$, is the set of isomorphism classes among $\{\mathcal{N}_r(v) \mid v \in V\}$.

PROBLEM 1. Given a positive integer r and a finite set Φ of rooted graphs, does there exist a graph G such that $\mathcal{N}_r(G)=\Phi$?

PROBLEM 2. Given positive integers r and c and a finite set Φ of rooted graphs, does there exist a graph G, having no cycles of length exceeding c, such that $\mathcal{N}_r(G)=\Phi$?

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In [7] Winkler showed that Problem 1 is undecidable and that Problem 2 is decidable. In both cases the result remains unchanged if *graph* is preceded by *connected.* Bulitko claims [2, p. 86] to have shown the undecidability of Problem 1 even when r is fixed to 1. However, Winkler's construction actually proved Problem 1's undecidability when the maximum vertex degree, d, is sufficiently large but bounded. Note that for fixed d and r there are only finitely many r-neighborhoods of maximum vertex degree d, and hence, up to isomorphism, only finitely many sets Φ. Thus Bulitko's reduction necessarily must have produced graphs of unbounded degree. Winkler's result and Bulitko's result, therefore, complement each other in that each author bounds one of these two parameters.

Recall that the *covering digraph* for a partial ordering $\mathscr{P} = (X, \leqslant)$ is the directed graph with vertex set X in which there is an edge from u to v if and only if v covers u (cf. [1]). (We say *v covers u* if $u < v$ and $u \leqslant w < v$ implies $u = w$.) When the covering digraph is represented on paper so that edges are oriented upward, the resulting figure is often called a *Hasse diagram* of $\mathscr{P}$ (cf. [3]). Given a finite set of local conditions for a partial ordering, we may wish to know if there exists a partial order on a finite set satisfying precisely these local conditions. We formulate this problem as follows, assuming r has been fixed.

PROBLEM 3. Given a finite set Φ of rooted digraphs, does there exist a (finite, partially ordered set with) covering digraph G satisfying $\mathscr{N}_r(G) = \Phi$?

When discussing digraphs, we use *connected*, *bipartite*, and *planar* if the properties hold in the underlying undirected graph. However, an *acyclic* digraph is one having no directed cycles. For graphs G with directed edges, colored vertices, and so forth, we assume that the members of $\mathscr{N}_r(G)$ also have the same attributes.

In this paper we show that Problem 3 *is undecidable for any* $r \geqslant 5$. *It remains undecidable, as does Problem* 1, *even if we insist that G is bipartite, connected, and planar.* To simplify our proof we define:

PROBLEM 4. Given finite sets Φ_1 and Φ_2 of rooted digraphs, does there exist a (finite partially ordered set with) covering digraph G satisfying $\Phi_1 \cap \mathscr{N}_r(G) \neq \varnothing$ and $\mathscr{N}_r(G) \subseteq \Phi_1 \cup \Phi_2$?

LEMMA 1. *Problem* 4 *is Turing-reducible to Problem* 3.

Proof. Let Φ_1 and Φ_2 be given. To decide this instance it suffices to consider each set Φ satisfying $\Phi_1 \cap \Phi \neq \varnothing$ and $\Phi \subseteq \Phi_1 \cup \Phi_2$ and to ask whether there exists a G with $\mathscr{N}_r(G) = \Phi$. Since there are only finitely many such Φ, the lemma holds. ∎

2. Context-Free Grammars

Now recall some basic ideas from formal language theory. A *context-free grammar* is a 4-tuple $\Gamma = (V, \Sigma, P, S)$ in which V is a finite set of *variables*, Σ is a disjoint, finite set of *terminals*, S (called the *start symbol*) is in V, and P is a finite subset of $V \times (V \cup \Sigma)^*$ (see [4]). As is customary, variables will be denoted with uppercase letters and terminals with lowercase letters. Lowercase Greek letters will denote *strings* (i.e., members of $(V \cup \Sigma)^*$). Members of P are called *productions* and are denoted as $A \to \sigma$, where $A \in V$ and $\sigma \in (V \cup \Sigma)^*$. Given a production $A \to \sigma$, and given strings τ, μ, we may write $\tau A \mu \Rightarrow \tau\sigma\mu$. We let $\overset{*}{\Rightarrow}$ denote the reflexive, transitive closure of $\Rightarrow$ on $(V \cup \Sigma)^*$. If $\sigma \overset{*}{\Rightarrow} \tau$ we say σ *derives* τ. A string $\sigma \in (V \cup \Sigma)^*$ is a *sentential form* of Γ if $S \overset{*}{\Rightarrow} \sigma$. The *language generated* by Γ, denoted $L(\Gamma)$, is $\{\sigma \in \Sigma^* \mid S \overset{*}{\Rightarrow} \sigma\}$. (That is, $L(\Gamma)$ is the set of sentential forms σ containing only terminals.) Figure 1 shows two grammars. The vertical bar symbol | means *or*, and so $Y \to b \mid Y$ is shorthand for the two productions $Y \to b$ and $Y \to Y$. It is well known (e.g., [4, p. 198]) that the following problem is undecidable.

Problem 5. Given context-free grammars Γ_1 and Γ_2, is $L(\Gamma_1) \cap L(\Gamma_2) \neq \varnothing$?

We will eventually show that Problem 4 is undecidable by reducing Problem 5 to it. Before doing so, we make several simplifying assumptions about Γ_1 and Γ_2:

(i) The right side of each production is a *nonempty* string.

(ii) The right side of each production is either a nonempty string of variables or a single terminal.

(iii) For each variable X that is not the start symbol, we have $X \to X$.

(iv) The start symbol never appears on the right side of a production.

$$
\begin{array}{llll}
\Gamma_1: & S_1 & \longrightarrow & L_1XYYR_1 \\
 & X & \longrightarrow & XY \mid a \mid X \\
 & Y & \longrightarrow & b \mid Y \\
 & L_1 & \longrightarrow & \ell \mid L_1 \\
 & R_1 & \longrightarrow & r \mid R_1 \\
 & & & \\
\Gamma_2: & S_2 & \longrightarrow & L_2UVR_2 \\
 & U & \longrightarrow & a \mid U \\
 & V & \longrightarrow & V \mid WWW \\
 & W & \longrightarrow & b \mid W \\
 & L_2 & \longrightarrow & \ell \mid L_2 \\
 & R_2 & \longrightarrow & r \mid R_2
\end{array}
$$

Fig. 1. Two context-free grammars.

Assumption (i) is justified since any context-free grammar Γ can be transformed into a context-free grammar Γ' having no empty productions and such that $L(\Gamma)=L(\Gamma')$ or $L(\Gamma)=L(\Gamma')\cup\{\varepsilon\}$ (see [4]). Hence, given Γ_1 and Γ_2, we can check if the empty string $\varepsilon\in\Gamma_1\cap\Gamma_2$. If not, we form Γ'_1, Γ'_2, and we know that $L(\Gamma_1)\cap L(\Gamma_2)=L(\Gamma'_1)\cap L(\Gamma'_2)$. Assumption (ii) is valid since, for each terminal a, we may replace its occurrences by an unused variable X_a, adding the production $X_a\to a$. Assumption (iii) is valid since we may add productions $X\to X$ for any variables X. Assumption (iv) is valid since we can redefine the start symbol S to be an unused variable S' and then add the production $S'\to S$.

We make one more modification to our grammars. Let L_1, R_1 be two unused variables, and let l, r be two unused terminals. Each production $S_1\to\sigma$, where S_1 is the start symbol of Γ_1, is now replaced by the production $S_1\to L_1\sigma R_1$. We also add the four productions $L_1\to L_1 \mid l$ and $R_1\to R_1 \mid r$. Similarly, we modify Γ_2 using new variables L_2, R_2 and the same terminals. If Γ'_1, Γ'_2, are the modified grammars, then clearly

$$L(\Gamma'_1)=\{l\sigma r \mid \sigma\in L(\Gamma_1)\}$$
$$L(\Gamma'_2)=\{l\sigma r \mid \sigma\in L(\Gamma_2)\}$$

and so $L(\Gamma_1)\cap L(\Gamma_2)\neq\varnothing$ if and only if $L(\Gamma'_1)\cap L(\Gamma'_2)\neq\varnothing$. Therefore we may assume

(v) Γ_1, Γ_2 have been modified with L_1, R_1, L_2, R_2, l, r as described above.

3. Decorated Digraphs

Throughout this paper, $\{\Gamma_1, \Gamma_2\}$ is an instance of Problem 5, *where $\Gamma_1=(V_1, \Sigma, P_1, S_1)$ and $\Gamma_2=(V_2, \Sigma, P_2, S_2)$ are context-free grammars satisfying conditions* (i)–(v). Without loss of generality, we assume $V_1\cap V_2=\varnothing$. The reduction of Problem 5 to Problem 4 will require mapping $\{\Gamma_1, \Gamma_2\}$ to an ordered pair (Φ_1, Φ_2), where Φ_i is a set of rooted digraphs. However, to simplify our mapping, we will first talk about *decorated* digraphs. By these we mean digraphs having "labeled" edges and "colored" vertices. Each edge in a decorated digraph will be labeled either *horizontal* or *vertical*. Each vertex in a decorated digraph will be colored with a grammar symbol from Γ_1 or Γ_2—either a variable or a terminal.

Decorated digraphs will be used to represent derivations in grammars. Consider the two grammars shown in Fig. 1. It turns out that *labbbr* is in $L(\Gamma_1)\cap L(\Gamma_2)$, and the decorated digraph in Fig. 2 makes this clear. The upper half of Fig. 2 shows that $S_1 \overset{*}{\Rightarrow} labbbr$ while the lower half of the figure shows that $S_2 \overset{*}{\Rightarrow} labbbr$. In our figures we assume that any edge

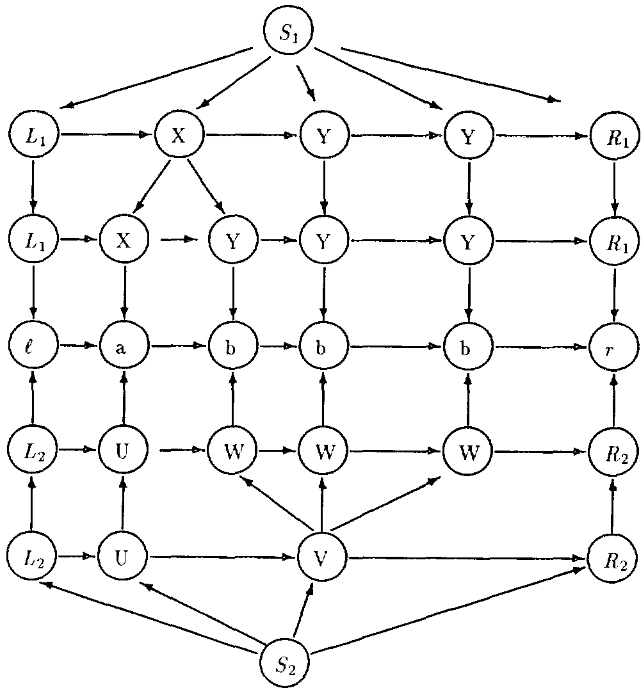

FIG. 2. A decorated graph showing $L(\Gamma_1) \cap L(\Gamma_2) \neq \varnothing$.

appearing as horizontal is a horizontal-labeled edge, while all other edges are vertical-labeled, even though some may actually appear slanted. Note that horizontal-labeled edges are oriented from left to right in Fig. 2, and paths of maximum length along horizontal-labeled edges yield sentential forms of the grammars. Vertical-labeled edges are used in productions.

4. DEFINING Φ_1 AND Φ_2

We now define two sets of rooted, decorated digraphs to be used throughout the remainder of this paper. The goal of this section is to establish the connection between $\{\Gamma_1, \Gamma_2\}$ and (Φ_1, Φ_2) given in Lemma 5 and Lemma 6. For each start production

$$S_1 \rightarrow L_1 X_{i_1} \cdots X_{i_t} R_1$$

we construct the decorated digraphs shown in Fig. 3, rooted at the S_1-colored vertex. Figure 3 actually represents a finite set of rooted, decorated digraphs having radius two. The vertices shown directly beneath X_{i_j} are

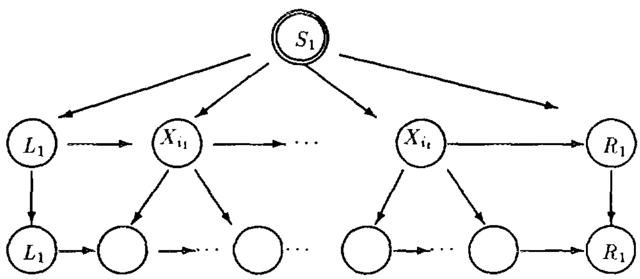

FIG. 3. The members of Φ_1.

colored according to all right sides of productions for X_{i_j}. In a similar way, we next construct all such rooted, decorated digraphs for productions of S_2. *Let Φ_1 be the set of all rooted, decorated digraphs constructed in this way.*

If p is the production $A \to \sigma$ we define $|p| = |\sigma| + 1$. Now let

$$M = \max\{|p| \mid p \in P_1 \cup P_2\}.$$

M is the maximum number of symbols, including the left side, in any production of Γ_1, Γ_2. Let

$$d = M + 2. \tag{1}$$

We now define Φ_2 to be the set of all rooted, decorated digraphs G satisfying:

(1) The root of G has color $\neq S_1, S_2$.

(2) G represents a 2-neighborhood.

(3) The degree of any vertex in G is $\leqslant d$.

(4) Vertices connected by a horizontal-labeled edge are both colored with terminals, or are both colored with variables of the same grammar.

(5) If the root is colored with L_i, then it has one incident horizontal edge which must be outgoing, and it has two incident vertical edges—one incoming and one outgoing. Its incoming vertical edge is incident with a vertex colored L_i or S_i. Its outgoing vertical neighbor must be colored L_i or l.

(6) If the root is colored with R_i, then it has one incident horizontal edge which must be incoming, and it has two incident vertical edges—one incoming and one outgoing. Its incoming vertical edge is incident with a vertex colored R_i or S_i. Its outgoing vertical neighbor must be colored R_i or r.

(7) If the root is colored with a terminal $t \neq l, r$, then it has two incident horizontal edges—one incoming and one outgoing, and two vertical edges both incoming.

(8) If the root is colored with l, then it has two incoming vertical edges—one incident with a L_1-colored vertex and the other incident with a L_2-colored vertex. It has one horizontal edge which must be outgoing.

(9) If the root is colored with r it has one (incoming) horizontal edge and two (both incoming) vertical edges.

(10) If the root is colored with a variable $A \neq L_i, R_i$ then it has two horizontal edges—one incoming and one outgoing.

(11) Finally, if the root is colored by a variable $A_1 \neq L_i$, then G must have an induced subgraph of the form shown in Fig. 4, where $A_0 \to B_{01} \cdots B_{0r}$ and $A_1 \to B_{11} \cdots B_{1t}$ are productions of the same grammar. The right side of these productions may be a single terminal. It is possible for A_0 to be an L_i, in which case its production would be $L_i \to L_i$ or $L_i \to l$. Similarly, A_1 could be an R_i.

By property (1), Φ_1 and Φ_2 are disjoint. Note that properties (2) and (3) ensure that Φ_2 is finite. The next three lemmas are illustrated in Fig. 5.

LEMMA 2. *Let G be a decorated digraph with $\mathcal{N}_2(G) \subseteq \Phi_1 \cup \Phi_2$ and let v_0 be an S_1-colored vertex of G. Then there exist two directed paths,*

$$v_0, v_1, ..., v_{n-1}, v_n$$

$$v'_0, v'_1, ..., v'_{m-1}, v'_m,$$

involving only vertical edges, where

1. *The vertices $\{v_i\}_{i=0}^n$ are all distinct as are $\{v'_i\}_{i=0}^m$.*
2. *$v'_m = v_n$ and is l-colored.*
3. *The paths intersect only at this one vertex.*
4. *Vertices v_i, $1 \leqslant i \leqslant n-1$, are L_1-colored. Vertices v'_i, $1 \leqslant i \leqslant m-1$, are L_2-colored.*
5. *v'_0 is colored S_2.*

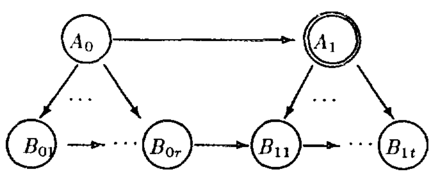

FIG. 4. Induced subgraph of 2-neighborhoods, $A_1 \neq L_i$.

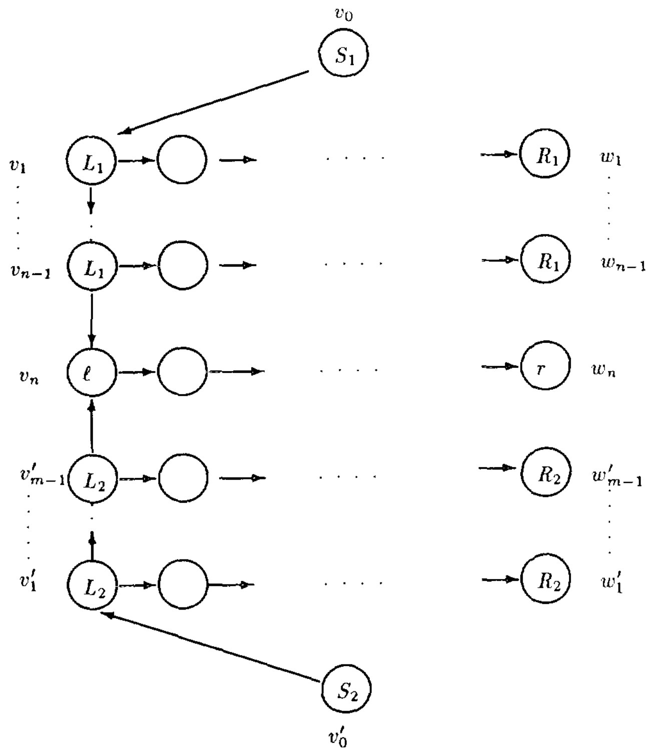

FIG. 5. Lemmas 2, 3, and 4 illustrated.

Proof. Note that by Fig. 3, v_0 has an outgoing vertical edge to a vertex, say v_1, colored L_1. Since $\mathcal{N}_2(v_1) \in \mathcal{N}_2(G) \subseteq \Phi_1 \cup \Phi_2$ and v_1 is not colored S_i, we have $\mathcal{N}_2(v_1) \in \Phi_2$. Thus $\mathcal{N}_2(v_1)$ must satisfy properties (1)–(11), and in particular, property (5). By (5), v_1 must have an outgoing vertical edge to a vertex v_2, colored either l or L_1. If v_2 is colored L_1, we again apply property (5) obtaining a vertex v_3. We continue this process until an l-colored vertex is found. Since v_0 has no incoming edges and each L_1-colored vertex has one incoming vertical edge, all vertices in the path are distinct. Therefore, by the finiteness of G, we must eventually terminate the path at an l-colored vertex v_n. Now using (8), there must be a L_2-colored vertex v' with a vertical edge *from* v' to v_n. We now use (5) again, repeatedly, working backwards until an S_2-colored vertex is found. We now call this vertex v'_0, the L_2-colored vertices $v'_1, ..., v'_{m-1} = v'$, and $v'_m = v_n$. ∎

LEMMA 3. *Let G be a decorated digraph with $\mathcal{N}_2(G) \subseteq \Phi_1 \cup \Phi_2$ and let v be an L_i-colored vertex of G. Then there exist an R_i-colored vertex w and a* *directed path of distinct vertices from v to w involving only horizontal-labeled edges and variable-colored vertices.*

Proof. By (5), there is an outgoing horizontal edge from v incident to another vertex. By (4), this vertex is colored with a member of V_i. If this color is R_i, then we are through. Otherwise, by (10), it has an outgoing horizontal edge to yet another variable-colored vertex. Continuing this process, we will never produce a cycle since every vertex has at most one incoming horizontal edge. By the finiteness of G, we must finally encounter a R_i-colored vertex. ∎

LEMMA 4. *Let G be a decorated digraph with $\mathcal{N}_2(G) \subseteq \Phi_1 \cup \Phi_2$, and let v be an l-colored vertex of G. Then there exist an r-colored vertex w and a directed path of distinct vertices from v to w involving only horizontal-labeled edges and terminal-colored vertices.*

Proof. The proof is similar to that of Lemma 3 but uses properties (8), (4), and (7). ∎

LEMMA 5. *If $L(\Gamma_1) \cap L(\Gamma_2) \neq \varnothing$ then there exists an acyclic, connected, planar, decorated digraph G such that $\Phi_1 \cap \mathcal{N}_2(G) \neq \varnothing$ and $\mathcal{N}_2(G) \subseteq \Phi_1 \cup \Phi_2$.*

Proof. If $\sigma \in L(\Gamma_1) \cap L(\Gamma_2)$ then we can represent the two derivations of σ using a decorated digraph G like the one shown in Fig. 2. Inspection of Fig. 2 shows that G is both connected and planar. It is also acyclic since any sufficiently long directed path will terminate at the unique r-colored vertex. To see $\Phi_1 \cap \mathcal{N}_2(G) \neq \varnothing$ and $\mathcal{N}_2(G) \subseteq \Phi_1 \cup \Phi_2$, it is easy to check that for both S_i-colored vertices v, we have $\mathcal{N}_2(v) \in \Phi_1$ and that for all other vertices v, $\mathcal{N}_2(v) \in \Phi_2$. ∎

LEMMA 6. *If there exists a decorated digraph G such that $\Phi_1 \cap \mathcal{N}_2(G) \neq \varnothing$ and $\mathcal{N}_2(G) \subseteq \Phi_1 \cup \Phi_2$, then $L(\Gamma_1) \cap L(\Gamma_2) \neq \varnothing$.*

Proof. Since $\Phi_1 \cap \mathcal{N}_2(G) \neq \varnothing$, G must have a 2-neighborhood of the kind in Fig. 3, and hence there must be a vertex v_0 colored S_1 or S_2. Without loss of generality, assume v_0 is colored S_1. Let

$$v_0, v_1, ..., v_{n-1}, v_n$$

$$v'_0, v'_1, ..., v'_{m-1}, v'_m$$

be the two sequences obtained from Lemma 2. By Lemma 3 and Lemma 4, we have for each v_i, v'_i, $i > 0$, horizontal paths terminating at an R_i-colored or r-colored vertex. This is shown in Fig. 5. For $i = 1, ..., n$ let σ_i be the string of symbols obtained by traversing the path from v_i to w_i. We claim that

$$S_1 = \sigma_0 \Rightarrow \sigma_1 \Rightarrow \sigma_2 \Rightarrow \cdots \Rightarrow \sigma_n.$$

To see this, first note that $\sigma_0 \Rightarrow \sigma_1$ from Fig. 3. When $1 \leqslant i \leqslant n-2$, we can show that $\sigma_i \Rightarrow \sigma_{i+1}$ as follows. Let

$$\sigma_i = L_i A_1 \cdots A_j R_i$$
$$\sigma_{i+1} = L_i B_1 \cdots B_k R_i.$$

Since $\mathcal{N}_2(G) \subseteq \Phi_1 \cup \Phi_2$, the A_1-colored vertex, by property (11), must have a 2-neighborhood containing an induced subgraph like the one shown in Fig. 4, and we must have $A_1 \Rightarrow B_1 \cdots B_t$ for some $t \leqslant k$. Applying Fig. 4 again, this time using the A_2-colored vertex as the root, we obtain $A_2 \Rightarrow B_{t+1} \cdots B_s$ for some $s \leqslant k$. Continuing, we eventually obtain $\sigma_i \Rightarrow \tau$ for τ an initial substring of σ_{i+1}. But here R_i can only derive R_i, and we must have $\sigma_i \Rightarrow \sigma_{i+1}$. A similar argument shows that $\sigma_{n-1} \Rightarrow \sigma_n$, except that σ_n contains terminals. This shows that $S_1 \stackrel{*}{\Rightarrow} \sigma_n$. Similarly, we can show that $S_2 \stackrel{*}{\Rightarrow} \sigma_n$ and so we have $\sigma_n \in L(\Gamma_1) \cap L(\Gamma_2)$, completing the proof. ∎

5. Transforming Decorated Digraphs to Covering Digraphs

In this section we show the undecidability of Problem 3. Let G be a decorated digraph. We first define a mapping that encodes G as an *undecorated* digraph G'. First, we replace each horizontal-labeled edge from

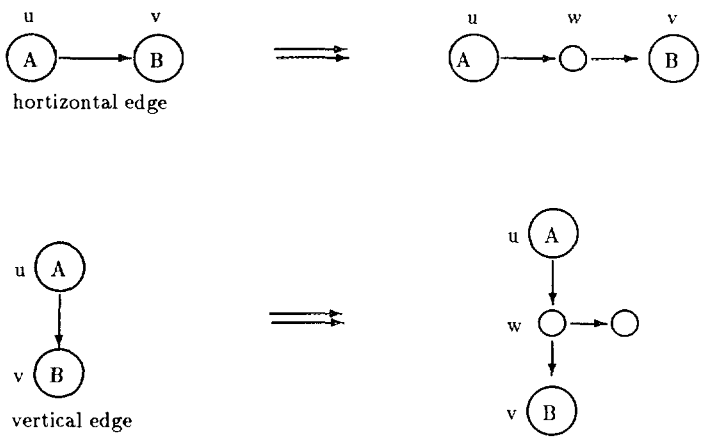

FIG. 6. Transforming horizontal- and vertical-labeled edges.

u to v by creating a new uncolored vertex w with an unlabeled edge from u to w and an unlabeled edge from w to v. For vertical edges we do the same, but also we construct an additional uncolored vertex incident with w. This operation is depicted in Fig. 6. Next, let d be the number defined in Eq. (1). We number all variables and terminals as

$$X_d, ..., X_{d+m}, \qquad a_{d+m+1}, ..., a_{d+m+n}.$$

We now replace each vertex colored with X_i or a_i with a degree i-vertex, adding if necessary, extra degree-1 vertices. We denote the resulting digraph by G'. This transformation is illustrated in Fig. 7. It is understood that if G is rooted, then G' is also rooted, having the "same" root. Note that if G has radius 2, then G' has radius 5. Of the following three lemmas, the first two are obvious, and the third is straightforward.

LEMMA 7. *Let G be a connected, planar, decorated digraph. Then G' is a connected, planar digraph.*

LEMMA 8. *Let G be an acyclic, decorated digraph. Then G' is acyclic.*

LEMMA 9. *A directed graph G is a covering digraph for a partial ordering if and only if*

(i) *G is acyclic.*

(ii) *If there exists a directed path from u to v having at least two edges, then there is no directed edge from u to v.*

LEMMA 10. *Let G be an acyclic, decorated digraph. Then G' is a covering digraph.*

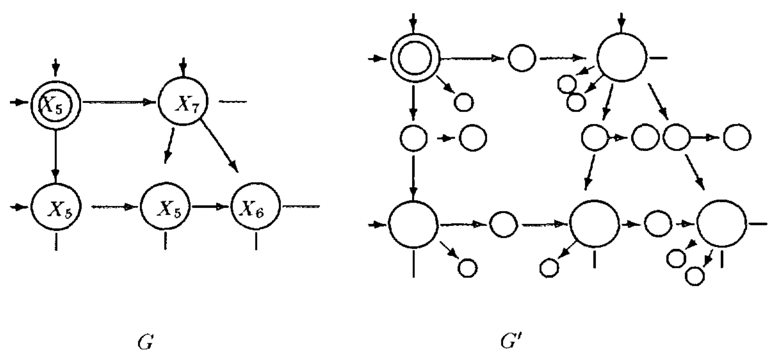

FIG. 7. Transforming a decorated digraph to a digraph.

Proof. It suffices to show that G' satisfies properties (i) and (ii) from Lemma 9. By Lemma 8, (i) holds. Also, (ii) holds because each new vertex w inserted between u and v, as shown in Fig. 6, will damage the possibility of a directed edge from u to v. ∎

LEMMA 11. *For any decorated digraph G, G' is bipartite.*

Proof. Note that the operation shown in Fig. 6 ensures that the resulting digraph is 2-colorable. The second operation, of adding degree-1 vertices, preserves 2-colorability, and so the resulting digraph is 2-colorable. ∎

Let Φ_1, Φ_2 be the sets of rooted, decorated digraphs constructed in Section 4. Recall that Φ_1 contains decorated digraphs with an S_i-colored root and that Φ_2 contains decorated digraphs with other-colored roots. In the following lemmas, Φ'_1 and Φ'_2 denote their images. It is obvious that $d \geqslant 4$, and so the root of each graph in $\Phi'_1 \cup \Phi'_2$ has degree at least 4.

We let Ψ be the set of all rooted digraphs having radius at most 5, vertex degrees at most $d+m+n$, with root having degree $\leqslant 3$. The set Ψ contains rooted digraphs used to accommodate the newly created degree-1, degree-2, and degree-3 vertices produced by our mapping.

LEMMA 12. *If $L(\Gamma_1) \cap L(\Gamma_2) \neq \varnothing$ then there exists a connected, bipartite, planar, covering digraph D such that $\Phi'_1 \cap \mathcal{N}_5(D) \neq \varnothing$ and $\mathcal{N}_5(D) \subseteq \Phi'_1 \cup \Phi'_2 \cup \Psi$.*

Proof. If $L(\Gamma_1) \cap L(\Gamma_2) \neq \varnothing$, then by Lemma 5 there exists an acyclic, connected, planar, decorated digraph G with $\Phi_1 \cap \mathcal{N}_2(G) \neq \varnothing$ and $\mathcal{N}_2(G) \subseteq \Phi_1 \cup \Phi_2$. By Lemma 7, Lemma 10, and Lemma 11, G' is a connected, bipartite, planar, covering digraph. Since 2-neighborhoods become 5-neighborhoods, it is easy to see that $\Phi'_1 \cap \mathcal{N}_5(G') \neq \varnothing$ and $\mathcal{N}_5(G') \subseteq \Phi'_1 \cup \Phi'_2 \cup \Psi$. ∎

LEMMA 13. *Let D be a connected digraph such that $\Phi'_1 \cap \mathcal{N}_5(D) \neq \varnothing$ and $\mathcal{N}_5(D) \subseteq \Phi'_1 \cup \Phi'_2 \cup \Psi$. Then there exists a connected, decorated digraph G such that $G' = D$.*

Proof (Sketch). For convenience, let us say that vertices of degrees 1, 2, or 3 have *small* degree and all other vertices have *large* degree. Since D is connected and has at least one large-degree vertex, using the definition of Φ'_1, an easy argument shows that all small-degree vertices are adjacent or distance two from some large-degree vertex.

We construct G as follows. First locate all large-degree vertices. Now color these vertices according to their degree, and delete any incident degree-1 vertices. At this point, all edges and uncolored vertices appear as in the right side of Fig. 6. For each pair of vertices u and v, as shown in

Fig. 6, replace the vertices and edges with a single edge from u to v, labeling it accordingly. ∎

LEMMA 14. *Let G be a connected, decorated digraph such that $\Phi'_1 \cap \mathcal{N}_5(G') \neq \varnothing$ and $\mathcal{N}_5(G') \subseteq \Phi'_1 \cup \Phi'_2 \cup \Psi$. Then $\Phi_1 \cap \mathcal{N}_2(G) \neq \varnothing$ and $\mathcal{N}_2(G) \subseteq \Phi_1 \cup \Phi_2$.*

Proof. Straightforward. ∎

LEMMA 15. *If there exists a digraph D satisfying $\Phi'_1 \cap \mathcal{N}_5(D) \neq \varnothing$ and $\mathcal{N}_5(D) \subseteq \Phi'_1 \cup \Phi'_2 \cup \Psi$, then $L(\Gamma_1) \cap L(\Gamma_2) \neq \varnothing$.*

Proof. We may assume that D is connected, since there must be some connected component satisfying the hypothesis. Let G be the connected, decorated digraph from Lemma 13 such that $G' = D$. By Lemma 14 we have $\Phi_1 \cap \mathcal{N}_2(G) \neq \varnothing$ and $\mathcal{N}_2(G) \subseteq \Phi_1 \cup \Phi_2$. By Lemma 6 we have $L(\Gamma_1) \cap L(\Gamma_2) \neq \varnothing$. ∎

THEOREM 1. *For each fixed $r \geqslant 5$, Problem 3 is undecidable.*

Proof. When $r = 5$, Lemma 12 and Lemma 15 show that

$$\text{Problem } 5 \leqslant_T \text{Problem } 4.$$

Here, Φ'_1 in Lemma 12 and Lemma 15 plays the role of Φ_1 in Problem 4, $\Phi'_2 \cup \Psi$ plays the role of Φ_2, and D plays the role of G. We have already observed in Lemma 1 that

$$\text{Problem } 4 \leqslant_T \text{Problem } 3.$$

Hence there is a Turing reduction from (the unsolvable) Problem 5 to Problem 3, establishing the theorem for $r = 5$. It is easy to show that if $r_1 < r_2$, then the r_1-problem is Turing-reducible to the r_2-problem. Hence Problem 3 is undecidable for *each* fixed $r \geqslant 5$. ∎

Since G' in Lemma 12 is connected, bipartite, and planar we also have

THEOREM 2. *For each fixed $r \geqslant 5$, Problem 3 remains undecidable if G is required also to be connected, bipartite, and planar.*

6. UNDIRECTED GRAPHS

Using a construction similar to the one above, we may transform decorated digraphs into (undirected) graphs. In order to encode directed edges as undirected edges, it seems necessary to now replace each directed edge with three undirected edges. Hence 2-neighborhoods become 7-neighborhoods.

THEOREM 3. *For each $r \geqslant 7$, Problem* 1 *remains undecidable when r is fixed and G is required also to be connected, planar, and bipartite.*

We now suggest two problems for future investigation. First, from the techniques in Lemma 3.2 of [5], it follows that Problem 5 is undecidable even when the number of productions is bounded by some sufficiently large integer. This suggests that possibly there exists an N such that Problem 1 and Problem 3 are undecidable even for the subproblems in which $|\Phi| \leqslant N$. Second, recall that Problem 2 is decidable [7]. Can we obtain a lower bound for this problem? Recall NEXPTIME is the class of languages recognizable by some nondeterministic Turing machine (NDTM) in time 2^{n^k} for some k. It is known that $\mathrm{NP} \subset \mathrm{NEXPTIME}$ (see [6]), and so any NEXPTIME-hard problem is not in NP, and consequently it cannot have short proofs. We conjecture that Problem 2 is nondeterministic exponential time-hard.

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